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S-rings on compact groups.

by G.V. Wood

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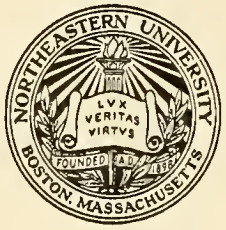

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MATH

S-rings on compact groups.

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1. Introduction

The study of closed subalgebras of the group algebra which are not ideals has been avoided for many years - even in the case of a compact abelian group. The main reason for this is the abundance of them. In this paper, we pick out a subclass of the closed subalgebras of C(G) - for a compact group G - which have considerably more structure, and which are consequently less numerous. These are the S-rings, which first appeared in finite groups ([2], [3], [6] et al.), but which provided the motivation for a generalization of the Peter-Weyl theorem in [7]. Here we give some structure theorems for S-rings and give the corresponding results to those of Tamaschke in [4] - in particular his homomorphism and isomorphism theorems.

2. Definitions and notation.

Let G be a compact topological group, and C(G) the set of complex-valued continuous functions on G. We shall use the notation

$$f^*(x) = \overline{f(x^{-1})} \; ; \; f^{\#}(x) = \overline{f(x)} \; ; \; (f*g)(x) = \int_G f(y)g(y^{-1}x)dm(y),$$

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where $m(\cdot)$ denotes Haar measure on G. Other notation will be as in [7].

DEFINITION 2.1 A subset $\mathcal{F}$ of C(G) is an S-ring on G if it is a pointwise subalgebra and a convolution subalgebra of C(G), * and $^{\#}$ self-adjoint, and contains the constant functions. (c.f. [6] p.56) Let $\mathcal{F}$ be an S-ring on G. We will denote by $\{G_\lambda : \lambda \in \Lambda\}$ its constancy sets. i.e. the equivalence classes of the equivalence relation ~ defined on G by

$$x \sim y \text{ if and only if } f(x) = f(y) \text{ all } f \in \mathcal{F}$$

We call a subset of G a Λ-set if it is the union of G_λ's and we call a function on G a Λ-function if it is constant on the G_λ's. Clearly, if $\mathcal{F}$ is a closed (= uniformly closed) S-ring, then, by the Stone-Weierstrass theorem, $\mathcal{F}$ is the set of all continuous Λ-functions. In fact $\mathcal{F} \cong C(\Lambda)$ as a commutative pointwise algebra.

In [7], we showed that the constancy sets of an S-ring satisfy the conditions: a) $G_\lambda G_\mu$ is a Λ-set (all $\lambda, \mu \in \Lambda$)

b) $(G_\lambda)^{-1} = G_\mu$ some $\mu \in \Lambda$

which imply c) if $1 \in G_{\lambda_0}$, G_{λ_0} is a subgroup and $G_\lambda = G_{\lambda_0} G_\lambda G_{\lambda_0}$ (all $\lambda \in \Lambda$)

It is not clear - though unlikely - that the conditions a) and b) are sufficient for the set of Λ-functions to form an S-ring. Even in the finite case, we need a stronger property than a),

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namely, that the elements of each $G_\nu \subset G_\lambda G_\mu$ should appear in $G_\lambda G_\mu$ with the same multiplicity. In the compact case, this condition is replaced by a measure theoretic condition which we will show is necessary and sufficient in §3. We shall use the notation $\overline{\mathcal{F}}^p$ for the closure of $\mathcal{F}$ in $L^p(G)$, for $p \geq 1$. Since $L^2(G) * L^2(G) \subset C(G)$, $L^1(G) * C(G) \subset C(G)$ and $L^p(G) \cdot C(G) \subseteq L^p(G)$, it is easy to see the following.

PROPOSITION 2.2 a) $\overline{\mathcal{F}}^2 * \overline{\mathcal{F}}^2 \subset \mathcal{F}$

b) $\overline{\mathcal{F}}^1 * \mathcal{F} \subset \mathcal{F}$

c) $\overline{\mathcal{F}}^p \cdot \mathcal{F} \subset \overline{\mathcal{F}}^p$.

3. Structure theorem.

Let $\mathcal{F}$ be a closed S-ring with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$.

LEMMA 3.1 If P is an open Λ-set, the characteristic function χ_P of P is in $\overline{\mathcal{F}}^2$.

Proof. Given $\varepsilon > 0$, since m is a regular measure, there exists a closed set $P_1 \subset P$ such that $m(P \setminus P_1) < \varepsilon$. By taking $\cup\{G_\lambda : G_\lambda \cap P_1 \neq \emptyset\}$ if necessary, we can assume that P_1 is a Λ-set. (This set is closed, since the G_λ's form an upper-semi-continuous (u.s.c.) family in G - see [1] p.125) By Urysohn's lemma in Λ, there exists $f \in \mathcal{F}$ such that $f(x) = 1$ on P_1, $f(x) = 0$ off P, $0 \leq f(x) \leq 1$ everywhere. Clearly $\| \chi_P - f \|_2^2 = \int_G |f(x) - \chi_P(x)|^2 \, dx \leq m(P \setminus P_1) < \varepsilon$. Thus $\chi_P \in \overline{\mathcal{F}}^2$.

THEOREM 3.2 Let $\mathcal{F}$ be a closed S-ring with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$. Then

a') For P, Q open Λ-sets, $m(P \cap xQ^{-1})$, as a function of x, is a continuous Λ-function.

b) $(G_\lambda)^{-1} = G_\mu$ - some μ.

Proof a') is obvious since $m(P \cap xQ^{-1}) = (\chi_P * \chi_Q)(x)$ and $\chi_P * \chi_Q \in \overline{\mathcal{F}}^2 * \overline{\mathcal{F}}^2 \subset \mathcal{F}$

b) is an easy consequence of $\mathcal{F}$ being *-self-adjoint.

Note that $m(P \cap xQ^{-1})$ corresponds to the "number of times" x appears in PQ. We will show that these conditions are sufficient. Let $\{G_\lambda : \lambda \in \Lambda\}$ be an upper-semi-continuous decomposition of G.

LEMMA 3.3 Any continuous Λ-function f can be approximated uniformly by step functions of the form $\sum_{i=1}^{n} a_i \chi_{A_i}$ where each A_i is the set theoretic difference of two open Λ-sets.

Proof. Suppose first that f is a non-negative function. Given $\varepsilon > 0$, there exists n such that $\|f\|_\infty < n\varepsilon$. Let

$$A_i = \{x : \tfrac{i}{n} \|f\|_\infty < f(x) \leq \tfrac{i+1}{n} \|f\|_\infty\} \quad i = 0, 1, 2, \ldots n-1$$

Then A_i is the difference of two open Λ-sets and it is easy to see that $\|f - \sum_{i=0}^{n-1} \frac{i}{n} \|f\|_\infty \chi_{A_i}\|_\infty < \varepsilon$. We deduce the result for general f, by writing $f = (\mathrm{Re}\ f)^+ - (\mathrm{Re}\ f)^- + i(\mathrm{Im}\ f)^+ - i(\mathrm{Im}\ f)^-$ which is the linear combination of positive Λ-functions.

THEOREM 3.4 Let $\{G_\lambda : \lambda \in \Lambda\}$ be an u.s.c. decomposition of G satisfying a') and b) of 3.2. Then the set $\mathcal{F}$ of continuous

Λ-functions forms an S-ring.

Proof. The only property not immediate is that $\mathcal{F}$ is a convolution algebra. If $A = P_1 \setminus P_2$, $B = Q_1 \setminus Q_2$ where $P_2 \subset P_1$, $Q_2 \subset Q_1$ are open Λ-sets, then

$$A \cap xB = (P_1 \cap xQ_1) \setminus (P_1 \cap xQ_2) \setminus [(P_2 \cap xQ_1) \quad (P_2 \cap xQ_2)].$$

Therefore $m(A \cap xB) = m(P_1 \cap xQ_1) - m(P_1 \cap xQ_2) - [m(P_2 \cap xQ_1) - m(P_2 \cap xQ_2)]$ which, as a linear combination of Λ-functions, is a Λ-function. Now if u and v are step functions defined by $u = \sum_{i=1}^{r} \alpha_j \chi_{A_i}$ and $v = \sum_{j=1}^{s} \beta_j \chi_{B_j}$ where A_i B_j, $i = 1 \ldots r$, $j = 1 \ldots s$ are differences of open Λ-sets, then $(u * v)(x) = \sum_{ij} \alpha_i \beta_j \, m(A_i \cap xB_j^{-1})$ and so $u * v$ is a Λ-function. But f and g can be uniformly approximated by step functions u_n and v_n respectively, and so $f * g$ is the uniform limit of the Λ-functions $u_n * v_n$, and hence is a Λ-function. This proves the theorem.

It is clear that a) and a') are not equivalent, though we have no example to show that a) and b) together are not equivalent to a') and b). Though clearly we have

$$G_\lambda G_\mu \text{ is a } \Lambda\text{-set (all } \lambda, \mu \in \Lambda) \iff \{x : m(P \cap xQ^{-1}) = 0\} \text{ is a } \Lambda\text{-set (all open } \Lambda\text{-sets } P, Q)$$

4. S-rings on subgroups.

A natural question to ask is whether the restrictions of the functions in $\mathcal{F}$ to a closed subgroup H also form an S-ring. The corresponding question for an arbitrary closed convolution

subalgebra of $C(G)$ is also of interest, but does not concern us here. We show that if H is a Λ-set, then the result is true. (This is trivial in the finite case.) We need two lemmas. Let H be a closed subgroup of G which is also a Λ-set.

LEMMA 4.1 The set $\mathcal{U}$ of all open Λ-sets U containing H which satisfy $U = HU$ form a base of neighbourhoods for H.

Proof. Let V be any open neighbourhood of H. Define $V' = \cup\{Hx : Hx \subset V\}$. Then V' is open since the right cosets of H form an u.s.c. decomposition of G, and $V' = HV'$. Let $V'' = \cup\{G_\lambda : G_\lambda \subset V'\}$. V'' is an open Λ-set, since the G_λ's are u.s.c. Finally let $U = HV''$, then U is an open Λ-set, $U = HU$ and $U = HV'' \subset HV' = V' \subset V$. The lemma is proved.

LEMMA 4.2 For each $U \in \mathcal{U}$, $f \in C(G)$, let $L_U f = \frac{1}{m(U)} \int_U f(y)dm(y)$. Then L_U is a linear functional on $C(G)$ and further the net $\{L_U : U \in \mathcal{U}\}$ has a weak limit L which satisfies

a) $L({}_x f) = Lf$ all $x \in H$

b) $Lf = 0$ whenever $f(x) = 0$ on H.

Proof. L_U is clearly a linear functional with $\|L_U\| \leq 1$. The family $\{L_U : U \in \mathcal{U}\}$ is weakly compact and so a subnet has a weak limit L. But $\mathcal{U}$ is a base of neighbourhoods and so every subnet tends weakly to L.

a) For $x \in H$, $L_U({}_x f) = \frac{1}{m(U)} \int_U f(x^{-1}y)dm(y) = \frac{1}{m(U)} \int_{xU} f(y)dm(y) = L_U f$ since $xU = U$. Passing to the limit, $L({}_x f) = f$.

b) Let $f \in C(G)$ with $f(x) = 0$ on H. Given $\varepsilon > 0$, $V = \{x : |f(x)| < \varepsilon\}$ is an open neighbourhood of H. By 4.1, there exists $U_0 \in \mathcal{U}$ such that $U_0 \subset V$. Therefore

$$|L_U f| \leq \frac{1}{m(U)} \int_U |f(y)| dy < \varepsilon \quad \text{all } U \in \mathcal{U},\ U \subset U_0.$$

Therefore $|Lf| \leq \varepsilon$, and since ε was arbitrary, $Lf = 0$.

COROLLARY 4.3 For $f \in C(H)$, define $L_0 f = Lf'$ where f' is any extension to G of f. Then L_0 is the linear functional on $C(H)$ corresponding to Haar measure on H.

Proof. L_0 is well-defined by b) and left-invariant by a). Further $L_0 1 = L1 = 1$ and so, by the uniqueness of Haar measure, L_0 is the Haar integral. (1 denotes the constant function)

THEOREM 4.4 If $\mathfrak{F}$ is a closed S-ring on G and H is a closed Λ-subgroup of G, the restrictions to H of the functions in $\mathfrak{F}$ form an S-ring on H.

Proof. The only non-trivial part of the proof is in showing that $\mathfrak{F}|_H$ is a convolution algebra. Since any continuous Λ-function on H can be extended to a continuous Λ-function on G (Tietze's theorem in Λ), it is sufficient to show that for $f, g \in \mathfrak{F}$, $f|_H * g|_H$ is a continuous Λ-function. Let $m_H(\cdot)$ denote Haar measure in H, and define h by $h(x) = \int_G f(y) g(y^{-1}x) dm_H(y)$. Then h is a continuous function on G, and is an extension of $f|_H * g|_H$. We will show that h is a Λ-function.

For $U \in \mathcal{U}$, define $h_U \in C(G)$ by

$$h_U(x) = \frac{1}{m(U)} \int_U f(y) g(y^{-1}x) dm(y) = \frac{1}{m(U)} (f\chi_U * g)(x).$$

By 4.2, $h_U(x) \to h(x)$ all $x \in G$. But, $\chi_U \in \overline{F}^2$ and so $f\chi_U * g \in F\overline{F}^2 * F \subset F$ (by 1.2 a) and c)). Therefore $h_U \in F$. Since h is the pointwise limit of A-functions, it is also a A-function. This completes the proof.

To prove the isomorphism theorem, we need the following theorem on ideals (= convolution ideals).

THEOREM 4.5 If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$ are all closed S-rings on G and $\mathcal{F}_2$ is an ideal of $\mathcal{F}_3$. Then $\mathcal{F}_1$ is an ideal of $\mathcal{F}_3$ if and only if $\mathcal{F}_1$ is an ideal of $\mathcal{F}_2$.

Proof. Obviously if $\mathcal{F}_1$ is an ideal of $\mathcal{F}_3$, then it is also an ideal of $\mathcal{F}_2$. To prove the converse we use the fact that S-rings have approximate identities ([7], Theorem 4.1). Let $f \in \mathcal{F}_1$, $g \in \mathcal{F}_3$. Then $f \in \mathcal{F}_2$ and so $f * g \in \mathcal{F}_2$. Let u_n be the approximate identity for f in $\mathcal{F}_1$. Then $u_n * f * g$ tends uniformly to $f * g$. But $u_n * f * g \in \mathcal{F}_1$ since $\mathcal{F}_1$ is an ideal in $\mathcal{F}_2$ and so $f * g \in \mathcal{F}_1$ since $\mathcal{F}_1$ is uniformly closed. This proves the theorem.

It is possible to generalise much of the theory of S-rings in finite groups - in particular the work of Tamaschke in [3] and [4]. It is pointless reproducing all his theorems since many of the proofs use essentially the same techniques. Here we shall simply pick out the more interesting results.

In fact, Tamaschke has considered two slightly different concepts: S-rings and S-semigroups*. The difference can be seen in the compact case in the conditions a) and a') in §3. With conditions a) and b), the set of all Λ-sets forms a semigroup with an identity and an involution, and the structure of this semigroup can be used in place of the S-ring structure to give almost the same algebraic results. But here we are more interested in the analytic results, and as the topology only has real significance in the case of S-rings, we shall consider that case here. In fact the results of [5] are proved for all groups and so, in particular, can be applied to our case. However, we shall remember that the Λ-sets do form a semigroup as this gives us a convenient way of defining a homomorphism in §6.

5. $\mathcal{F}$-normal subgroups.

Let K be a closed subgroup of G, and denote by $\mathcal{F}_K$ the S-ring of all continuous functions on G which are constant on the double cosets of K. We give the generalisation of "normal" subgroup as in [4], remembering that K is normal if and only if $\mathcal{F}_K$ is an ideal of C(G).

Let $\mathcal{F}$ be a closed S-ring with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$. Then $\mathcal{F} \cap \mathcal{F}_K$ is also an S-ring, and if K is a Λ-set, then its

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* Originally, Tamaschke considered double-coset S-rings and double-coset semigroups ([3]), but recently these results have been extended to the general case ([4] and [5]).

constancy sets are $\{KG_\lambda K : \lambda \in \Lambda\}$. We will denote $\mathcal{F} \cap \mathcal{F}_K$ by $\mathcal{F}/_K$. As in [4], we have:

LEMMA 5.1 If K is a Λ-subgroup, the following are equivalent.

i) $\mathcal{F}/_K$ is a convolution ideal of $\mathcal{F}$

ii) $G_\lambda K = KG_\lambda$ all $\lambda \in \Lambda$.

Proof. Assuming ii), the constancy sets of $\mathcal{F}/_K$ are $\{KG_\lambda : \lambda \in \Lambda\}$. For $f \in \mathcal{F}/_K$, $g \in \mathcal{F}$, $f * g \in \mathcal{F}$ and $f * g$ will be constant on the right cosets of K (since f is). Therefore $f * g$ is constant on all KG_λ's hence $f * g \in \mathcal{F}/_K$. Similarly for $g * f$, and so $\mathcal{F}/_K$ is an ideal of $\mathcal{F}$.

Conversely, suppose $\mathcal{F}/_K$ is an ideal of $\mathcal{F}$. Then if $G_\lambda \cap G_\mu K = \emptyset$, by normality, there exists an open neighbourhood U of K such that $G_\lambda U \cap G_\mu U = \emptyset$, and we can assume that U is a union of double cosets of K and a Λ-set. (as in 4.1) Now there exists $f \in \mathcal{F}/_K$ with $f(x) = 1$ on K, $f(x) = 0$ off U^{-1}, $0 \leq f(x) \leq 1$ everywhere, and $g \in \mathcal{F}$ with $g(x) = 1$ on $G_\mu K$, $g(x) = 0$ off $G_\mu U$, $0 \leq g(x) \leq 1$ everywhere. Then $(g * f)(x) = 0$ for $x \in G_\lambda$ since $G_\mu U \cap xU = \emptyset$ and $(g * f)(y) \neq 0$ for $y \in G_\mu$. But $g * f \in \mathcal{F}/_K$ and hence $\mathcal{F}/_K$ separates the sets $\{G_\lambda K : \lambda \in \Lambda\}$. Therefore $G_\lambda K = KG_\lambda K = KG_\lambda$.

DEFINITION 5.2. If K is a Λ-subgroup satisfying i) and ii) of 5.1, then K is $\mathcal{F}$-normal.

REMARK: If $\mathcal{F} = C(G)$, we have the usual definition of a normal subgroup and the usual property $xK = Kx$. (all $x \in G$)

6. Homomorphism and Isomorphism theorems.

Our next aim is to define an S-ring homomorphism. At first sight, the natural definition would be to make $\varphi: \mathcal{F} \to \mathcal{F}'$ an algebra homomorphism with respect to both multiplications and both involutions. However, this already makes φ a 1 - 1 map and isometric in $\|\cdot\|_1$ and $\|\cdot\|_2$. Instead we use the algebraic definition as in [5], which is an obvious generalisation of group homomorphism. We will see that this definition of homomorphism is sufficient for the homomorphism and isomorphism theorems, but that the corresponding isomorphism does not have the desirable property of being a convolution isomorphism. For this reason, we do not use the term 'S-ring-homomorphism', but 'CS-homomorphism'. (CS = constancy set)

Let $\mathcal{F}$ be a closed S-ring with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$. By condition a) of §3, there is a "multiplication" in Λ, where the "product" of two elements in Λ is a subset rather than a single element of Λ. Precisely, there is a map of $\Lambda \times \Lambda$ into the subsets of Λ defined by $\lambda.\mu = \{\nu : G_\nu \subset G_\lambda G_\mu\}$. There is also an involution in Λ defined by $\lambda^* = \mu$ where $(G_\lambda)^{-1} = G_\mu$. If $1 \in G_{\lambda_0}$, we have the identities

$$\lambda.\lambda_0 = \lambda_0.\lambda = \{\lambda\} \quad \text{all } \lambda \in \Lambda \; ; \; \lambda_0^* = \lambda_0.$$

Let $\mathcal{F}$ and $\mathcal{F}'$ be closed S-rings on G and G' respectively, with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$ and $\{G'_\omega : \omega \in \Omega\}$.

DEFINITION 6.1 φ is a CS-homomorphism of $\mathcal{F}$ to $\mathcal{F}'$ if φ is a continuous map $\Lambda \to \Omega$ satisfying:

a) $\varphi(\lambda.\mu) = (\varphi\lambda).(\varphi\mu)$ (all $\lambda, \mu \in \Lambda$)

b) $\varphi(\lambda^*) = (\varphi\lambda)^*$ (all $\lambda \in \Lambda$)

c) $\varphi(\lambda_0) = \omega_0$

By defining $\varphi(P) = \cup\{G'_{\varphi\lambda} : G_\lambda \subset P\}$ for each Λ-set P, we can think of φ as a map of the semigroup of Λ-sets into the semigroup of Ω-sets. Then the conditions of the definition imply that φ is a semigroup homomorphism satisfying $\varphi(P^{-1}) = (\varphi P)^{-1}$ and $\varphi(UP_i) = \cup(\varphi P_i)$ where P, P_i, $i = 1, 2 \ldots$ are Λ-sets. (c.f.[5]) It will be convenient to use the symbol φ for either of these maps.

REMARK. Although the map φ is not defined on $\mathcal{F}$, but on Λ, there is an induced map $\tilde{\varphi}: C(\Omega) \to C(\Lambda)$ defined by

$$(\tilde{\varphi}f)(\lambda) = f(\varphi\lambda)$$

This, in turn, induces a continuous map of $\mathcal{F}'$ into $\mathcal{F}$, which is an algebra homomorphism with respect to pointwise multiplication, and is isometric in the uniform norm. This gives us yet another way of defining a homomorphism, but we will not go into the details here.

DEFINITION 6.2 The kernel of a CS-homomorphism φ is defined by

$$\ker \varphi = \varphi^{-1}(G'_{\omega_0}) = \cup\{G_\lambda : \varphi\lambda = \omega_0\}$$

DEFINITION 6.3 The Image of a CS-homomorphism φ is defined by

$$\text{Im } \varphi = \varphi(G) = \cup\{G'_\omega : \omega \in \varphi\Lambda\}$$

Clearly CS-monomorphism, epimorphism, isomorphism can be defined in the usual way.

THEOREM 6.4 (Homomorphism theorem.) Let $\mathcal{G}$ and $\mathcal{H}$ be closed S-rings on G and G' respectively with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$ and $\{G'_\omega : \omega \in \Omega\}$. If φ is a CS-homomorphism, then

a) $\ker \varphi$ is an $\mathcal{G}$-normal subgroup

b) Im φ is a Ω-subgroup

c) $\mathcal{G}/_{\ker \varphi}$ is CS-isomorphic to $\mathcal{H}|_{\text{Im } \varphi}$

Proof. a) Let $K = \ker \varphi$. Then certainly $K^{-1} = K$, and for each $\omega \in \varphi\Lambda$, $\varphi^{-1}(G'_\omega) = \cup\{G_\lambda : \varphi\lambda = \omega\}$ is a closed Λ-set. Also $\varphi(KG_\lambda K) = (\varphi K)(\varphi G_\lambda)(\varphi K) = G'_{\omega_0} G'_\omega G'_{\omega_0} = G'_\omega$. Therefore $KG_\lambda K \subset \varphi^{-1}(G'_\omega)$ (all λ with $\varphi\lambda = \omega$). Conversely, for fixed λ with $\varphi\lambda = \omega$, suppose μ also satisfies $\varphi\mu = \omega$. Then $\varphi(\lambda^*.\mu) = (\varphi\lambda)^*.\mu = \omega^*.\omega \ni \omega_0$. Therefore $\lambda^*.\mu \cap \varphi^{-1}(\omega_0) \neq \emptyset$ and so $\mu \in \lambda.(\varphi^{-1}\omega_0)$ i.e. $G_\mu \subset G_\lambda K$. Therefore $KG_\lambda K \subset \varphi^{-1}(G'_\omega) \subset G_\lambda K$. Thus $\varphi^{-1}(G'_\omega) = KG_\lambda K = G_\lambda K = KG_\lambda$ (all λ such that $\varphi\lambda = \omega$). In particular $KK = K$, and since $K^{-1} = K$, K is a subgroup. By 5.1, K is an $\mathcal{G}$-normal subgroup.

b) Since $\varphi(G) = \varphi(GG) = (\varphi G)(\varphi G)$ and $(\varphi G)^{-1} = \varphi(G^{-1}) = \varphi(G)$, $H' = \text{Im } \varphi$ is a closed Ω-subgroup of G'.

c) By 4.4, $\mathcal{H}|_{H'}$ is an S-ring with constancy sets $\{G_\omega : \omega \in \Phi\Lambda\}$. By above, $\{\varphi^{-1}(G'_\omega) : \omega \in \Phi\Lambda\}$ are the constancy sets of $\mathcal{G}/_K$ and the identity on $\varphi\Lambda$ induces a CS-isomorphism of $\mathcal{G}/_K$ with $\mathcal{H}|_{H'}$.

THEOREM 6.5 (1st Isomorphism theorem) $\mathcal{F}$ is a closed S-ring on G and K is an $\mathcal{F}$-normal subgroup of G. If L is any other subgroup:

L is $\mathcal{F}$-normal if and only if L is $\mathcal{F}/_K$ normal.

This is a direct corollary of 3.6 using the three S-rings $\mathcal{F}$, $\mathcal{F} \cap \mathcal{F}_K$, and $\mathcal{F} \cap \mathcal{F}_K \cap \mathcal{F}_L$.

THEOREM 6.6 (2nd Isomorphism theorem) $\mathcal{F}$ is a closed S-ring on G with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$. If K is an $\mathcal{F}$-normal subgroup and L is a Λ-subgroup then

a) $K \cap L$ is $\mathcal{F}|_L$-normal

b) $KL = LK$ and K is $\mathcal{F}|_{KL}$-normal

c) $\mathcal{F}|_L/_{K\cap L}$ is CS-isomorphic to $\mathcal{F}|_{KL}/_K$.

This proof follows immediately for 4.4, 5.1 and 6.4.

The surprising part of this section is that the homomorphism and isomorphism theorems are true when the homomorphism is essentially the algebraic definition corresponding to S-semigroups. To define S-ring-homomorphism, we would have to strengthen condition a) of Definition 6.1 to read

a') $m_G(P \cap x Q^{-1}) = m_{\varphi G}(\varphi P \cap y(\varphi Q)^{-1})$ where P, Q are Λ-sets, $x \in G_\lambda$ and $y \in G'_{\varphi\lambda}$. This would then give us an S-ring-isomorphism which <u>is</u> an isomorphism with respect to convolution. Again, no example is known to distinguish between the two ideas.

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